Inner Product Spaces

Definition 1. *An* ***inner product space*** *is a vector space with an inner product $\langle v, w \rangle$. In this note the vector space is assumed to be real.*

An inner product induces a norm, $\|v\| = \sqrt{\langle v, v \rangle}$, and hence a distance $d(v, w) = \|v - w\|$. We can speak about Cauchy sequences and define convergence, like we did in $\mathbb{R}^n$. A **complete inner product space** is one in which every Cauchy sequence converges. $\mathbb{R}^n$ with the Euclidean inner product is complete (Bolzano-Weierstrass). We've essentially proved that any other inner product on $\mathbb{R}^n$ defines an inner product space that is isomorphic in a precise sense to the Euclidean inner product space.

Definition 2. *A* ***Hilbert Space*** *is a complete inner product space.*

One of our favorite inner product spaces is $\mathcal{R}([a, b])$ with inner product

$$\langle f, g \rangle = \int_a^b fg.$$

$\mathcal{R}([a, b])$ is **not** a Hilbert space. It's a theorem that if H_1, H_2 are Hilbert spaces that are not finite dimensional and have countable dense sets then they are isomorphic. So for example it might not be true that two inner products on the same vector space are equivalent.

Here's an example. Let H be $\mathcal{L}^2([a, b])$ (with inner product $\langle f, g \rangle = \int_a^b fg$). As a vector space H is huge, but it does have a countable dense set, and it is complete. It has a vector space basis $\{v_\alpha\}$, called a Hamel basis, that has the cardinality of the continuum [1]. Every element v of H has a unique expression as a finite linear sum $v = \sum_j v_j$. Define a new inner product on H by

$$\langle v, w \rangle_q = \sum_j v_j w_j,$$

A distance can be defined with this inner product

$$d_q(v, w) = \|v - w\|_q .$$

In this norm H doesn't have a countable dense set. The distance between any two basis vectors $d_q(v_\alpha, v_\beta) = \sqrt{2}$. Since there are uncountably many basis vectors, the two inner products on the same space, H, are not equivalent. They define different topologies (open sets).

References

[1] Tsing, Nam-Kiu, Infinite-Dimensional Banach Spaces Must Have Uncountable Basis–An Elementary Proof, *The American Mathematical Monthly*, Vol. 91, No. 8 (Oct., 1984), pp. 505-506